validity for our life indeed is sufficiently proven by *experience*, whose *general, necessary exactness* however could be doubted without absurdity."

Alas! that this feeble utterance should have seemed sufficient for more than thirty years to the associate of Gauss and Schweikart, the latter certainly one of the independent discoverers of the non-Euclidean geometry. But then since neither of these sufficiently realized the transcendent importance of the matter to publish any of their thoughts on the subject, a more adequate conception of the issues at stake could scarcely be expected of the scholar and colleague. How different with Bolyai János and Lobachévski, who claimed at once, unflinchingly, that their discovery marked an epoch in human thought so momentous as to be unsurpassed by anything recorded in the history of philosophy or of science, demonstrating as had never been proven before the supremacy of pure reason at the very moment of overthrowing what had forever seemed its surest possession, the axioms of geometry.

*Austin, Texas, December 16th, 1895.*

# THE DUPLICATION OF THE NOTATION FOR IRRATIONALS.

By JOS. V. COLLINS, Ph. D., State Normal School, Stevens Point, Wisconsin.

Authorities agree in crediting Rudolff (1525), the German cossist, with the introduction of the radical sign, $\sqrt{}$, not precisely as we use it, but one such mark for a square root, three for a cube, and two for a fourth root. Cantor thinks it probably originated from a West-Arabian custom of using dots, by makings *lines* of the dots, and connecting them in the making by lighter lines. These dots in turn originated, it is thought, in the use of the letter, dschim, the first in the Arabian word for *root*. Rudolff was followed by Stifel in the employment of this notation, and afterwards Girard (1633) changed it to the present form. By the middle of the 17th century the mark had come into general use. The exponent notation, though first used by Rudolff and Stifel in a crude form, was employed as we now have it for integral values of the exponents by Descartes. Soon after, Wallis, in his *arithmetica infinitorum* (1656), interpreted and used the simpler forms of fractional exponents, though Stevin (1585) had suggested the meaning to be assigned them. Then in 1676 Newton wrote to Oldenburg "since algebraists write $a^2$, $a^3$, $a^4$, etc., for $aa$, $aaa$, $aaaa$, etc., so I write $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$, for $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{e.a^5}$." Newton went further in connection with his binomial theorem, and generalized this use of exponents into the exponential function. The question naturally arises why was it that the old notation for roots was not replaced by the new as had been done in numerous instances before? Doubtless the best

reason for this is the fact that the radical signs were firmly intrenched by extended use before the fractional exponents as we have them were even thought of.

Now from one standpoint at least this duplication of marks for one of the commonest operations in mathematics is unfortunate. It certainly complicates unnecessarily a rather difficult part of elementary algebra. Doubtless all would agree that one or the other should be given up unless there is a good and sufficient reason for its retention. If either is to be discarded there is no question for a moment as to which should go. The use of fractional exponents is in perfect accord with that of integral ones, and introduces no new marks or conventions, while the radical sign notation is out of harmony with everything else in the algebraic notation. The radical sign and index are new marks, while the fractional exponent is an old quantity in a new place whose interpretation is quite natural. However, it should be said that the fractional exponent notation is ambiguous, since, in general,

$\left( a^m \right)^{\frac{1}{n}}$ will not be the same as $\left( a^{\frac{1}{n}} \right)^m$, though each reduces to $a^{\frac{m}{n}}$. Nevertheless, even here the fractional exponent notation is to be preferred to the others, since the elementary treatment of irrationals virtually depends on the ignoring of this difference. (See, for example, Todhunter's Algebra, ed. 1877, p. 153; Chrystal's Treatise, Chapter X, Part II.) Not a few authors succeed by their manner of treatment in slurring this over. In this connection it ought to be said that some authors' books show distinct traces of their having been confused by the double surd notation. If authors themselves are not clear in their treatment of irrationals, it is likely that their students also will be more or less puzzled. This of itself would be a sufficient justification of an effort to remove the difficulty.

One obstacle in the way of dispensing entirely with the radical signs consists in the practical difficulty of writing and printing fractional exponents. But this, one is constrained to believe, can readily be overcome. And first it may be remarked that there is the same justification for omitting the numerator 1 in a fractional exponent that there is for never writing the integral exponent 1. When omitted it can be understood. Then again there is the same justification for dropping the denominator 2 in the exponent that there is for understanding the radical index 2 when no index is written. Thus all that is left of the fractional exponent $\frac{1}{2}$ is the horizontal line or the solidus oblique line. To make the changes suggested clear to the reader, some expressions are written below with their values in the three notations:

| RADICAL NOTATION. | | FRAC. EXPONENT NOT. | | PROPOSED NOTATION. |
|---|---|---|---|---|
| $2\sqrt{a}$ | $=$ | $2a^{\frac{1}{2}}$ | $=$ | $2a^{/}$ * |

*The marks for primes would differ from this sign in being shorter and vertical. However, it would be better to write subscripts in place of them.

$$
\begin{array}{ccccc}
3\sqrt[3]{26} & = & 3(26)^{3} & = & 3(26)^{/3} \\
\sqrt{a+m} & = & (a+m)^{\frac{1}{2}} & = & (a+m)^{/} \\
\sqrt[4]{\dfrac{a^2+b^2+c^2}{2abc}} & = & \left(\dfrac{a^2+b^2+c^2}{2abc}\right)^{\frac{1}{4}} & = & \left(\dfrac{a^2+b^2+c^2}{2abc}\right)^{/4} \\
\sqrt[5]{(x^3+3xy^2)^3} & = & \multicolumn{3}{c}{(x^3+3xy^2)^{\frac{3}{5}}\text{, or }(x^3+3xy^2)^{3/5}}
\end{array}
$$

The proposed notation would do away with vinculums and would use preferably the solidus sign for division as is the tendency now in English mathematical and scientific books. In printing, $\sqrt{\phantom{x}}/$ would be replaced by $)/$ on one type, and in script the latter would be made, without lifting the pen, in loop form. However, when the numerator of the fractional exponent is other than unity, the usual fractional exponent notation (which for this case is preferable to the radical sign notation) would be employed. Notice that by the simple changes proposed, which are perfectly natural ones, all the advantages of the duplicate notation would be preserved with none of its disadvantages, such as the use of the unsightly hieroglyphic-like radical sign (giving as it does a forbidding appearance to the printed page), and the confusion which arises from the simultaneous use of two distinct notations for the same operation.

In conclusion it should be emphasized that mathematicians themselves are not likely to feel the need or approve of any change in the algebraic notation. Like the reform in spelling, it is in the interest chiefly of the hundreds of thousands of students of elementary mathematics yet to come, and not in that of those who have already mastered the two notations, that this reform is urged. Surely it is not too much to ask that the fractional exponents as now written be employed exclusively (instead of largely as now) in all higher works involving the use of algebraical symbols. The abridgments would then be likely to come as a matter of course.

*Stevens Point, Wisconsin, May 11, 1895.*

# INTRODUCTION TO SUBSTITUTION GROUPS.

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[Continued from December Number.]

## THE CONSTRUCTION OF NON-PBIMITIVE GROUPS WITH THREE SYSTEMS OF NON-PRIMITIVITY.

Let the degree of the required group $G$ be $3n$. $G$ must be a subgroup (using subgroup in its broad sense in wnich it includes the group itself and identity) of